# Risk Neutral Processes for Market Curve Evolution in the Vector Risk System

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## 1 RNFX Process: Risk Neutral Stochastic Process for FX Spot Rates

An instantaneous risk neutral stochastic process for an FX spot exchange rate $S(t)$ between two currencies CCY1 and CCY2 (the single rate point on the FX-PRICE-CCY1-CCY2 curve) takes the following form:

$$d\,\mathrm{ln}S = (F_0^1(t) - F_0^2(t))\,dt + \sigma dX \tag{1.0.1}$$

where

- $dX = \phi\sqrt{dt}$ where $\phi = \mathcal{N}(0,1)$ (*i.e.*, a Wiener process).
- $\sigma$ is the implied volatility.
- $F_0^i(t)$ is the instantaneous forward rate for currency $i = 1, 2$ applying to the (future) period $(t, t+dt)$ as observed today (at time $t = 0$).

The instantaneous forward rate is defined to be:

$$F_0^i(t) \equiv \lim_{\delta t \to 0} f_0^i(t, t + \delta t) \tag{1.0.2}$$

where $f_0^i(t_1, t2)$ is defined to be the forward rate, as observed today (at time $t = 0$) for currency $i$ applicable to the time interval $(t_1, t_2)$, which is given by:

$$f_0^i(t_1, t_2) = \frac{t_2 R_0^i(t_2) - t_1 R_0^i(t_1)}{t_2 - t_1} \tag{1.0.3}$$

and where $R_0^i(t)$ is the today zero rate applicable to the period $(0, t)$. It follows that:

$$\begin{aligned}
F_0^i(t) &= \lim_{\delta t \to 0} \frac{(t + \delta t)R_0^i(t + \delta t) - tR_0^i(t)}{\delta t} && (1.0.4)\\
&= R_0^i(t) + t \lim_{\delta t \to 0} \frac{R_0^i(t + \delta t) - R_0^i(t)}{\delta t} && (1.0.5)\\
&= R_0^i(t) + t\frac{dR_0^i}{dt}(t) && (1.0.6)\\
&= \frac{d}{dt}\left[tR_0^i(t)\right] && (1.0.7)
\end{aligned}$$

In order to use this process in credit exposure simulations, we must solve for the distribution of $S(t)$ at some future time $t$, given its value at time $t_0 \geq 0$ $S(t_0)$ is known, where $t > t_0$. We do so by first finding a solution to the deterministic equation:

$$\mathrm{ln}S = (F_0^1(t) - F_0^2(t))\,dt \tag{1.0.8}$$

viz:

$$\mathrm{ln}S_{\mathrm{det}}(t) = \mathrm{ln}S(t_0) + \int_{t_0}^{t} (F_0^1(s) - F_0^2(s))\,ds \tag{1.0.9}$$

Now, using (1.0.7), we obtain:

$$\begin{aligned}
\int_{t_0}^{t} F_0^i(s)\,ds &= \left[sR_0^i(s)\right]_{t_0}^{t} && (1.0.10)\\
&= tR_0^i(t) - t_0 R_0^i(t_0) && (1.0.11)\\
&= (t - t_0) f_0^i(t_0, t) && (1.0.12)
\end{aligned}$$

whence, the solution to the deterministic equation is:

$$\ln S_{\rm det}(t) = \ln S(t_0) + (t - t_0)\left[f_0^1(t_0, t) - f_0^2(t_0, t)\right] \tag{1.0.13}$$

We can then use variation of parameters to solve the full stochastic equation by looking for a solution that adds an arbitrary constant to the deterministic solution that we allow to be deterministic, *viz*:

$$\ln S(t) = \ln S_{\rm det}(t) + c \tag{1.0.14}$$

where $c$ is stochastic and must satisfy:

$$dc = \sigma\, dX \tag{1.0.15}$$

Now, the general solution to the stochastic differential equation

$$dc = h(t)\, dX \tag{1.0.16}$$

is

$$c = \Sigma(t_0, t)\phi + C \tag{1.0.17}$$

where $\phi = \mathcal{N}(0, 1)$ and $C$ is an arbitrary, deterministic constant (independent of $t$) and:

$$\Sigma(t_0, t) \equiv \sqrt{\int_{t_0}^{t} (h(s))^2\, ds} \tag{1.0.18}$$

In the case of interest we have $h(s) = \sigma$, and so:

$$\Sigma(t_0, t) = \sigma\sqrt{t - t_0} \tag{1.0.19}$$

Now, to satisfy the initial condition $\lim_{t\to t_0} S(t) = S(t_0)$, we must choose $C = 0$, and so the solution for the distribution of $S(t)$ is:

$$\ln S(t) = \ln S(t_0) + (t - t_0)\left[f_0^1(t_0, t) - f_0^2(t_0, t)\right] + \Sigma(t_0, t)\phi \tag{1.0.20}$$

where $\phi$ is a deviate drawn from $\mathcal{N}(0, 1)$.

## 1.1 Generalisation to Non-Constant Volatility

The risk neutral FX process can be generalised to a non-constant volatility version, *viz*:

$$d\ln S = (F_0^1(t) - F_0^2(t))\, dt + \sigma_{\rm IF}(t) dX \tag{1.1.1}$$

where $\sigma_{\rm IF}$ is now the instantaneous forward volatility, defined as follows:

$$\sigma_{\rm IF}(t) \equiv \lim_{\delta t\to 0} \sigma_{\rm F}(t, t + \delta t) \tag{1.1.2}$$

with

$$\sigma_{\rm F}(t_1, t_2) \equiv \sqrt{\frac{t_2\sigma^2(t_2) - t_1\sigma^2(t_1)}{t_2 - t_1}} \tag{1.1.3}$$

and where the quantity $\sigma(t)$ denotes a maturity-dependent (non-constant) volatility.

To solve for the distribution of $S(t)$, we again employ the variation of parameters approach (1.0.14). The deterministic part of the stochastic process is unchanged here and so $\S_{\rm det}$ once more satisfies (1.0.13). The equation satisfied by the stochastic component of $S$ in this case is:

$$dc = \sigma_{\rm IF}(t) dX \tag{1.1.4}$$

which again takes the form (1.0.16) with

$$h(t) = \sigma_{\rm IF}(t) \tag{1.1.5}$$

and so in this case, the solution for the distribution is once again (1.0.20), but with

$$\Sigma(t_0, t) = \sqrt{\int_{t_0}^{t} (\sigma_{\rm IF}(s))^2\, ds} \tag{1.1.6}$$

Now, using (1.1.2) and the definition (1.1.3), we can write

$$\sigma_{\mathrm{IF}}(t) = \lim_{\delta t\to 0}\sqrt{\frac{(t+\delta t)\sigma^2(t+\delta t)-t\sigma^2(t)}{\delta t}} \tag{1.1.7}$$

$$= \sqrt{\lim_{\delta t\to 0}\sigma^2(t+\delta t)+t\lim_{\delta t\to 0}\frac{\sigma^2(t+\delta t)-\sigma^2(t)}{\delta t}} \tag{1.1.8}$$

$$= \sqrt{\sigma^2(t)+t\frac{d}{dt}\sigma^2(t)} \tag{1.1.9}$$

$$= \sqrt{\frac{d}{dt}\left(t\sigma^2(t)\right)} \tag{1.1.10}$$

and so (1.1.6) simplifies to:

$$\Sigma(t_0,t) = \sqrt{\int_{t_0}^{t}\frac{d}{ds}\left(s\sigma^2(s)\right)\,ds} \tag{1.1.11}$$

$$= \sqrt{\left[s\sigma^2(s)\right]_{t_0}^{t}} \tag{1.1.12}$$

$$= \sqrt{t\sigma^2(t)-t_0\sigma^2(t_0)} \tag{1.1.13}$$

$$= \sigma_{\mathrm{F}}(t_0,t)\sqrt{t-t_0} \tag{1.1.14}$$

## 1.2 RNFX Process Implementation Details and Calibration

The implied volatility term structure $\sigma(t)$ used in the RNFX process will be captured from the FX-PRICEVOL-CCY1-CCY2 curve, assuming it exists. (If a delta-based volatility surface is being used for the currency pair of interest, the FX-PRICEVOL-CCY1-CCY2 curve will exist as a component of the surface (the ATM volatility term structure component), in addition to other curves such as risk reversals and strangles. The RNFX process will just use the ATM component in such cases, and ignore the other components. By default, the zero curves that define the $R_0^i(t)$ for the two currencies in the RNFX process will be of type FX-ZERO-CCY1-RESCCY and FX-ZERO-CCY2-RECCCY (the cross-currency basis curves), where CCY1 and CCY2 are the currencies of interest and RESCCY is the reserve currency, currently a global assumption hardcoded to USD. If these curves are not present, then standard proxies will generally default back to the standard discounting curves for the two currencies: MM-ZERO-SWAP-CCY1 and MM-ZERO-SWAP-CCY2.

# 2 HWAUTO Process: Risk Neutral Hull-White Zero Curve Stochastic Process Definition

## 2.1 Definition of the HW Process

The single-factor HW model with constant short rate volatility and reversion factor is given by the following stochastic process:

$$dr = (\theta(t)-ar)\,dt+\sigma\,dX \tag{2.1.1}$$

where:

- $r = r(t)$ denotes the short-term (instantaneous) interest rate at future time $t \geq 0$.
- $r(0) \equiv r_0$ is the known short rate at $t = 0$, taken to be the shortest term rate (*e.g.*, the overnight rate) from today's known term structure. Here we are assuming that the yield curve has flat extrapolation outside of the range of prescribed terms on the curve.
- $dX = \phi\sqrt{dt}$ where $\phi = \mathcal{N}(0,1)$ (*i.e.*, a Wiener process).
- $a$ and $\sigma$ are stochastic parameters for the short rate. $a$ is the mean reversion rate and $\sigma$ is the short term volatility.
- $\theta(t)$ is a function chosen to fit the known $t = 0$ term structure (*i.e.*, to reproduce today's ZCB prices), for any given values of $a$ and $\sigma$.
- The parameters $a$ and $\sigma$ are usually determined by fitting market Swaption and/or Caplet prices to analytical results from the model.

The probability distribution for the short rate $r(t)$ at all times is normal. This makes the model very tractable.

The model can be generalised so that $a$ and/or $\sigma$ are functions of time. This allows Caplet and/or Swaption prices to be fitted more closely. However, as pointed out by Hull and White, this can in fact lead to mispricing of other derivatives and their recommendation was to keep the parameters independent of time. Different practitioners have different views on this, and Hull no longer seems to be against having at least a time-dependent volatility. Having the parameters time-dependent makes the equations more complicated. In BR so far we don't support time-dependent $a$ and/or $\sigma$. In fact, $a$ is currently taken to be a trade field for HW products such as Bermudan Swaptions, though this should at least be a modelling parameter. I believe that we have seen some evidence to suggest that the fitting function for $a$ and $\sigma$ can show weak dependence on the overall size of $a$ and $\sigma$, and strong dependence on the ratio. As a result, there may be some utility in fixing $a$ at a "sensible" value rather than allowing $a$ and $\sigma$ to both vary and then arrive at very high or low values that give a slightly better fit than a sensible value. The main reason for fixing $a$ however is to simplify the HW lattice and calibration implementations and to get substantial algorithmic performance improvements. In calibrating a HW model for a given currency for the purposes of risk-neutral CVA simulations, we can generalise our fitting market prices to analytical Swaption and/or Cap prices to allow $a$ to vary in the calibration and then use that value of $a$ as the modelling assumption value for products that use the HW lattice.

## 2.2 Definition of Key Quantities for Which We Need to Solve

In order to define $\theta(t)$ and the HW expressions for forward ZCB prices, *etc.*, we require some definitions:

- $R(t,T)$ is the continuously compounded zero rate that will apply at some future time $t \geq 0$ for the period $(t,T)$, where $T \geq t$. $R_0(T) \equiv R(0,T)$ is today's zero rate applicable to the term $T$. Note that $r(t) = R(t,t)$ and $r_0 = R_0(0)$.

- $P(t,T)$ is the future price at time $t \geq 0$ of a ZCB paying \$1 at (maturity) time $T \geq t$. $P_0(T) \equiv P(0,T)$ is today's discount factor applicable to the term $T$, *viz*:

$$P_0(T) \equiv P(0,T) = e^{-R_0(T)T} \tag{2.2.1}$$

  In general we have the future discount factor:

$$P(t,T) = e^{R(t,T)(T-t)} \tag{2.2.2}$$

- $f(t,T_1,T_2)$ is the continuously compounded forward rate as seen at future time $t \geq 0$ for contracts starting at time $T_1 \geq t$ and ending at $T_2 \geq T_1$. The forward rate in general satisfies:

$$f(t,T_1,T_2) = \frac{\ln\left[P(t,T_1)\right] - \ln\left[P(t,T_2)\right]}{T_2 - T_1}$$

  At $t = 0$ we have the usual relations:

$$f(0,T_1,T_2) = \frac{\ln\left[P_0(T_1)\right] - \ln\left[P_0(T_2)\right]}{T_2 - T_1} = \frac{R_0(T_2) - R_0(T_1)}{T_2 - T_1}$$

  which generalise to

$$f(t,T_1,T_2) = \frac{\ln\left[P(t,T_1)\right] - \ln\left[P(t,T_2)\right]}{T_2 - T_1} = \frac{R(t,T_2) - R(t,T_1)}{T_2 - T_1} \tag{2.2.3}$$

- $F(t,T)$ is the instantaneous forward rate at future time $t \geq 0$ for contracts maturity at time $T \geq t$, *viz*:

$$F(t,T) \equiv \lim_{\Delta T \to 0} f(t,T,T+\Delta T) = -\frac{\partial}{\partial T} \ln P(t,T)$$

  We thus have:

$$F_0(T) \equiv F(0,T) = -\frac{\partial}{\partial T} \ln P_0(T) = R_0(T) + R_0'(T)T \tag{2.2.4}$$

  and:

$$F_0'(T) = 2R_0'(T) + R_0''(T)T \tag{2.2.5}$$

## 2.3 Choice of $\theta(t)$ to Match Initial Term Structure

The choice of the $\theta(t)$ function required in order to get the HW model to fit the initial term structure is given by:

$$\theta(t) = F_0'(t) + aF_0(t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \tag{2.3.1}$$

Using (2.2.4) and (2.2.5) we can obtain $\theta(t)$ directly from the term structure and its numerical derivatives. In the BR rate cache it was our intention to implement a function that returns $R_0'(T)$, possibly making use of the piecewise linear functional form (*i.e.*, linear interpolation) that is generally used for yield curves in BR. The HW models used in BR require the calculation of $R_0'(T)$ but not $R_0''(T)$. That is, $R_0'(T)$ is required for the calculation of ZCB prices as part of the calibration to Swaptions and Caps in order to determine $\sigma$ and $a$, but this does not in fact require the calculation of $\theta(t)$. Neither is it necessary to compute $\theta$ in the HW lattice implementation for, *e.g.*, Bermudan pricing, as the condition of consistency of the model with the initial term structure is an integrated part of the lattice construction procedure which doesn't require a knowledge of $\theta$. In the absence of a particular BR rate cache function to compute $R_0'(T)$, it is computed by a simple finite difference, *viz*:

$$R_0'(T) \approx \frac{R_0(T + \Delta T) - R_0(T)}{\Delta T} \tag{2.3.2}$$

where $\Delta T \equiv 1/365$.

We do in principle require $\theta(t)$ in order to implement an MC simulation of future yield curve movements, *e.g.*, in the context of CVA calculations. We could probably create rate cache functionality in order to return an estimate of $R_0''(T)$, *e.g.*, by fitting a quadratic through the closet three points on the term structure to $T$, the term of interest.

However, since today's term structure is taken to be a piecewise linear function (*i.e.*, a discrete set of values are defined and then extended by linear interpolation), it makes sense to define $\theta(t)$ in a similar way. That is, we define $\theta(t)$ on the same set of prescribed term points as the yield curve and extend by interpolation if necessary, *e.g.*, by assuming $\theta(t)$ is piecewise constant. Let $\{t_i : i = 1, \ldots, n\}$ be the $n$ term points on which the yield curve is defined, with corresponding yields $R_0(t_1), \ldots, R_0(t_n)$. For a given term $t_i$ we fit a quadratic through the three closest points on the yield curve: $(t_{i-1}, R_0(t_{i-1}))$, $(t_i, R_0(t_i))$ and $(t_{i+1}, R_0(t_{i+1}))$, *viz*:

$$f(t) = \alpha(t - t_{i-1})(t - t_i) + \beta(t - t_i)(t - t_{i+1}) + \gamma(t - ti - 1)(t - t_{i+1}) \tag{2.3.3}$$

with:

$$\begin{aligned} \alpha &= \tfrac{R_0(t_{i+1})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \\ \beta &= \tfrac{R_0(t_{i-1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} \\ \gamma &= \tfrac{R_0(t_i)}{(t_i-t_{i-1})(t_i-t_{i+1})} \end{aligned} \tag{2.3.4}$$

We then have:

$$f'(t_i) = \alpha(t_i - t_{i-1}) + \beta(t_i - t_{i+1}) + \gamma(2t_i - t_{i+1} - t_{i-1}) \tag{2.3.5}$$

$$f''(t_i) = 2(\alpha + \beta + \gamma) \tag{2.3.6}$$

We can then respectively identify $R_0'(t_i)$ and $R_0''(t_i)$ with these expressions for $f'(t_i)$ and $f''(t_i)$ to calculate the expressions (2.2.4) and (2.2.5) to define $F_0(t_i)$ and $F_0'(t_i)$ respectively, and thence to compute the $\theta(t_i)$, the values of the $\theta$ function on the prescribed terms, using (2.3.1). For the end points $t_1$ and $t_n$ we can simply assume linear forms $f(t) = R_0(t_1) + \frac{R_0(t_2)-R_0(t_1)}{t_2-t_1}(t - t_1)$ and $f(t) = R_0(t_n) + \frac{R_0(t_n)-R_0(t_{n-1})}{t_n-t_{n-1}}(t - t_n)$ respectively whence we have, at $t = t_1$: $R_0'(t_1) = \frac{R_0(t_2)-R_0(t_1)}{t_2-t_1}$ and $R_0''(t_1) = 0$; and at $t = t_n$, $R_0'(t_n) = \frac{R_0(t_n)-R_0(t_{n-1})}{t_n-t_{n-1}}$ and $R_0''(t_n) = 0$.

A quadratic fit of the closest three points on the yield curve could also be used as an alternative method of computing $R_0'(T)$ to the abovementioned numerical derivative approach. After obtaining the quadratic fit, the fitting function can be differentiated to obtain the required derivative $R_0'(T)$. Such function for the first and second derivative of the term structure could be added to the NormalCurve class so that they are generally available.

We shall see that, as it turns out, when the HW stochastic process is solved for the distribution function for the short rate $r(t)$, it is possible to avoid the need for directly calculating $\theta$ (and hence for the direct calculation of $R_0''(T)$). This is because the solution of the SDE for $r(t)$ requires only the integral of $\theta(t)$ over $t$ rather than $\theta(t)$ itself. A calculation of $\theta$ would only be required if the stochastic process for the short rate were to be integrated numerically (iterated explicitly in small time steps). Such an implementation of the HW stochastic process would serve as a useful check of the analytic solution of the SDE.

## 2.4 HW Expression for ZCB Prices and Other Key Quantities of Interest

The HW model expression for the future ZCB price is given by:

$$P(t, T) \equiv P(r(t)|t, T) = A(t, T)e^{-B(t,T)r(t)} \tag{2.4.1}$$

where:

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \tag{2.4.2}$$

and:

$$\begin{aligned}
\ln A(t,T) &= \ln\frac{P_0(T)}{P_0(t)} + B(t,T)F_0(t) - \frac{\sigma^2}{4a^3}\left(e^{-aT} - e^{-at}\right)^2\left(e^{2at} - 1\right) && (2.4.3)\\
&= \ln\frac{P_0(T)}{P_0(t)} + B(t,T)F_0(t) - \frac{\sigma^2}{4a}\left(1 - e^{-2at}\right)B(t,T)^2 && (2.4.4)\\
&= R_0(t)t - R_0(T)T + B(t,T)F_0(t) - \frac{\sigma^2}{4a}\left(1 - e^{-2at}\right)B(t,T)^2 && (2.4.5)
\end{aligned}$$

Since $A(t,T)$ and $B(t,T)$ are deterministic, the only stochastic dependence in the ZCB prices is through the short rate in the exponential. Since the short rate is normally distributed, the future ZCB prices are lognormally distributed. This makes the valuation of Bond Options, and hence European Swaptions, relatively straightforward.

Inverting the relation (2.2.2) and using (2.4.1), we obtain the HW expression for the future zero rate applying at time $t$ to the term $T - t$:

$$R(t,T) \equiv R(r(t)|t,T) = -\frac{\ln P(t,T)}{T-t} = \frac{1}{T-t}\left[-\ln A(t,T) + B(t,T)r(t)\right] \tag{2.4.6}$$

We thus note that in the HW model, interest rates applying to any given term at all future times are normally distributed, and a knowledge of the future short term rate, (*e.g.*, in the context where its value is simulated from the stochastic equation), allows us to compute the term structure, ZCB prices and forward rates at that time. Using (2.2.3) and (2.4.6) we obtain:

$$\begin{aligned}
f(t,T_1,T_2) \equiv f(r(t)|t,T_1,T_2) &= \frac{R(r(t)|t,T_2) - R(r(t)|t,T_1)}{T_2 - T_1} && (2.4.7)\\
&= \left[\frac{B(t,T_1)}{T_1 - t} - \frac{B(t,T_2)}{T_2 - t}\right]r(t) + \ln\frac{A(t,T_2)}{A(t,T_1)} && (2.4.8)
\end{aligned}$$

whence future forward rates are also normally distributed. This makes the valuation of Caps relatively straightforward.

## 2.5 Probability Distribution for the Short Rate

The HW stochastic process (2.1.1) is a fairly standard mean-reverting process and the distribution function can be derived using the variation of parameters approach. In general we wish to solve for $r$ at some time $t$ given a knowledge of $r$ at some earlier time $t_0 < t$. We start by solving for $r(t)$ in the absence of the stochastic term, *viz*:

$$dr = (\theta(t) - ar)\,dt \tag{2.5.1}$$

or

$$\frac{dr}{dt} + ar = \theta(t) \tag{2.5.2}$$

Multiplying by $e^{at}$ we obtain:

$$\frac{d}{dt}re^{at} = e^{at}\theta(t) \tag{2.5.3}$$

whence

$$r(t) = ce^{-at} + H(t) \tag{2.5.4}$$

with $c$ some constant and:

$$H(t) \equiv H(t_0,t) \equiv e^{-at}\int_{t_0}^{t} e^{as}\theta(s)\,ds \tag{2.5.5}$$

We then look for a solution to the full, stochastic equation (2.1.1) of the form (2.5.4) where we now take $c$ to be a stochastic quantity. Differentiating (2.5.4), we have:

$$\begin{aligned}
dr &= e^{-at}\,dc - ae^{-at}c\,dt + H'(t)\,dt && (2.5.6)\\
&= e^{-at}\,dc - ar\,dt + aH(t)\,dt + H'(t)\,dt && (2.5.7)
\end{aligned}$$

where we have transformed (2.5.4) to obtain $ce^{-at} = r(t) - H(t)$. Also, from the definition (2.5.5) we have:

$$H'(t) = -aH(t) + \theta(t) \tag{2.5.8}$$

and so we can rewrite (2.5.7) as

$$dr = e^{-at}\, dc + (\theta(t) - ar)\, dt \tag{2.5.9}$$

Identifying with (2.1.1), we then have:

$$dc = \sigma e^{at}\, dX \tag{2.5.10}$$

Now, the general solution to an equation of the form:

$$dx = u(t)\, dX \tag{2.5.11}$$

is

$$x(t) \sim \mathcal{N}\left(x(t_0), \sqrt{\int_{t_0}^{t} (u(s))^2\, ds}\right) \tag{2.5.12}$$

whence

$$\begin{aligned} c(t) &\sim \mathcal{N}\left(c(t_0), \sigma\sqrt{\int_{t_0}^{t} e^{2as}\, ds}\right) && (2.5.13)\\ &= \mathcal{N}\left(c(t_0), \sigma\sqrt{\int_{t_0}^{t} e^{2as}\, ds}\right) && (2.5.14)\\ &= \mathcal{N}\left(c(t_0), \frac{\sigma e^{at}}{\sqrt{2a}}\sqrt{1-e^{-2a(t-t_0)}}\right) && (2.5.15)\end{aligned}$$

Substituting back into (2.5.4) we then have:

$$r(t) \sim \mathcal{N}\left(e^{-at}c(t_0) + H(t), \frac{\sigma}{\sqrt{2a}}\sqrt{1-e^{-2a(t-t_0)}}\right) \tag{2.5.16}$$

Since $H(t_0) = 0$ we can solve for $c(t_0)$ to obtain:

$$r(t) \sim \mathcal{N}\left(e^{-a(t-t_0)}r(t_0) + H(t), \frac{\sigma}{\sqrt{2a}}\sqrt{1-e^{-2a(t-t_0)}}\right) \tag{2.5.17}$$

or

$$r(t) = \mu(t_0, t) + \Sigma(t_0, t)\phi \tag{2.5.18}$$

where $\phi$ is a uniform deviate drawn from $\mathcal{N}(0, 1)$

$$\mu(t_0, t) \equiv e^{-a(t-t_0)}r(t_0) + H(t_0|t) \tag{2.5.19}$$

and

$$\Sigma(t_0, t) \equiv \frac{\sigma}{\sqrt{2a}}\sqrt{1-e^{-2a(t-t_0)}} \tag{2.5.20}$$

are the mean and standard deviation respectively.

Since $\theta(t)$ and hence $H(t_0|t)$ are deterministic quantities, this confirms that $r(t)$ is normally distributed and hence that all future zero rates $R(t, T)$ are also normally distributed, with the distributions obtained by substituting (2.5.17) into (2.4.6).

The function $H(t_0|t)$ depends on $\theta$ which in turn depends on the second derivative of the initial term structure. However, we can simplify as follows:

$$\int_{t_0}^{t} F_0'(s)e^{as}\, ds = [F_0(s)e^{as}]_{t_0}^{t} - a\int_{t_0}^{t} F_0(s)e^{as}\, ds \tag{2.5.21}$$

$$\int_{t_0}^{t} [F_0'(s) + aF_0(s)]e^{as}\, ds = F_0(t)e^{at} - F_0(t_0)e^{at_0} \tag{2.5.22}$$

$$\begin{aligned} \int_{t_0}^{t} [1-e^{-2as}]e^{as}\, ds &= \frac{2}{a}[\cosh(as)]_{t_0}^{t} && (2.5.23)\\ &= \frac{2}{a}[\cosh(at) - \cosh(at_0)] && (2.5.24)\end{aligned}$$

Combining these results with the defintions (2.3.1) and (2.5.5), we get:

$$H(t_0|t) = F_0(t) - F_0(t_0)e^{-a(t-t_0)} + \frac{\sigma^2 e^{-at}}{a^2}[\cosh(at) - \cosh(at_0)] \tag{2.5.25}$$

and thus, substituting into (2.5.19), we have

$$\mu(t_0, t) = e^{-a(t-t_0)}r(t_0) + F_0(t) - F_0(t_0)e^{-a(t-t_0)} + \frac{\sigma^2 e^{-at}}{a^2}[\cosh(at) - \cosh(at_0)] \tag{2.5.26}$$

## 2.6 Analytic Solutions for Zero Coupon Bond Options and Options on Coupon Bearing Bonds

Formulae for pricing ZCB Options, Bonds and Bond Options are a precursor to pricing instruments that are used to calibrate the stochastic parameters of HW model.

### 2.6.1 ZCB Options

We consider an Option on a ZCB with nominal amount $L$, strike $K$ where the option matures at time $t_1$ and the bond matures at time $t_2$. We denote the price of the ZCB Option by:

$$V_{\mathrm{ZCBO}}(L, K, t_1, t_2, \varphi) \tag{2.6.1}$$

where:

$$\varphi = \begin{cases} + & \text{Call} \\ - & \text{Put} \end{cases} \tag{2.6.2}$$

The pricing formulae are given by:

$$V_{\mathrm{ZCBO}}(L, K, t_1, t_2, +) = LP_0(t_2)N(h) - KP_0(t_1)N(h - \sigma_P) \tag{2.6.3}$$

$$V_{\mathrm{ZCBO}}(L, K, t_1, t_2, -) = KP_0(t_1)N(-h + \sigma_P) - LP_0(t_2)N(-h) \tag{2.6.4}$$

where:

$$\sigma_P \equiv \frac{\sigma\left(1 - e^{-a(t_2 - t_1)}\right)\sqrt{\frac{\left(1 - e^{-2at_1}\right)}{2a}}}{a} \tag{2.6.5}$$

$$h \equiv \frac{\ln\left(\frac{L}{K}\right) + r_1 t_1 - r_2 t_2}{\sigma_P} + \frac{\sigma_P}{2} \tag{2.6.6}$$

### 2.6.2 Coupon Bond Options

We consider a Coupon Bond with Nominal $S$, Coupon Rate $c$, Tenor $\tau$ (expressed in years), and Start and End Dates $T_{\mathrm{s}}$ and $T_{\mathrm{e}}$ (expressed in years from the valuation date).

The associated collection of cash flows $(L_1, L_2, \ldots, L_N)$ that are paid at times $(T_1, T_2, \ldots T_N)$ satisfy:[1]

$$L_i = c\tau L \quad 1 \le i < N \tag{2.6.7}$$

$$L_N = (1 + c\tau)L \tag{2.6.8}$$

$$T_1 \approx T_{\mathrm{s}} + \tau \tag{2.6.9}$$

$$T_i \approx T_{i-1} + \tau \quad 1 < i < N \tag{2.6.10}$$

$$T_N = T_{\mathrm{e}} \approx T_{N-1} + \tau \tag{2.6.11}$$

The price at some time $t$ of a Coupon Bond $V_{\mathrm{CB}}$ is simply the sum of the present values of the cashflows:

$$V_{\mathrm{CB}}(L, c, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, t) \equiv V_{\mathrm{CB}}(r(t)|c, K, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, t) = \sum_{i=1}^{N} L_i P(r(t)|t, T_i) \tag{2.6.12}$$

At $t = 0$ this reduces to the deterministic expression:

$$V_{\mathrm{CB}}(L, c, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, K, 0) = \sum_{i=1}^{N} L_i P_0(T_i) \tag{2.6.13}$$

To calculate the value of an option on a coupon bearing bond with option maturity $T$ and strike $K$, we first calculate the short rate $r_K$ that would make the value of the underlying bond equal to the strike. *I.e.*, we find $r_K = r_K(L, c, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau)$ such that[2]:

$$\sum_{i=1}^{N} L_i P(r_K|T, T_i) = K \tag{2.6.14}$$

[1]The approximately equals signs indicate that the payment/fixing dates have to be adjusted according to the business day convention so that they don't fall on non-business days.

[2]Since all of the $L_i$ are positive and assuming the strike $K$ is positive and doesn't exceed $\sum_{i=1}^{n} L_i$, this equation has a unique solution as $P(r(t)|t, T) = A(t, T)e^{-B(t,T)r(t)}$ is a monotone function of $r(t)$. The solution is readily obtained using, *e.g.*, Newton-Raphson or Brent's Method

For each cashflow in the underlying bond we create an option on a ZCB where the strike is the cashflow priced using $r_K$. The value of the Coupon Bond Option $V_{\mathrm{CBO}}$ is the sum of the value of all of these ZCB Options.

$$V_{\mathrm{CBO}}(L, c, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, K, \varphi, T) = \sum_{i=1}^{N} V_{\mathrm{ZCBO}}(L_i, K_i, T, T_i, \varphi) \tag{2.6.15}$$

where:

$$K_i \equiv K_i\left(L, c, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, T\right) \equiv L_i P\left(r_K | T, T_i\right) \tag{2.6.16}$$

and, once again, $\varphi = +$ and $\varphi = -$ denotes Puts and Calls respectively.

### 2.6.3 Analytic Prices of Swaptions and Caplets/Floorlets

We can now price the key calibrating instruments for the HW model: Swaptions and Caps/Caplets. We note that in the discussion so far and what follows, we are assuming an Act/365 Day Count Convention on accruals.

- **Swaptions** We consider a Swaption where the underlying Swap has Strike Rate $s_K$ (*i.e.*, the rate paid on the fixed leg), Nominal $L$, Start Date $T_{\mathrm{s}}$, End date $T_{\mathrm{e}}$, and Tenor $\tau$. We take the Expiry (Exercise) date of the Swaption to be equal to the Start Date $T_{\mathrm{s}}$ of the underlying Swap.

  The value of a Swaption $V_{\mathrm{s}}$ is the same as the Option on a Bond where the Coupon is equal to the swaption strike and the Bond Option Strike is equal to the Nominal, *viz*:[3]

$$V_{\mathrm{s}}\left(L, s_K, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, \varphi\right) = V_{\mathrm{CBO}}(L, s_K, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, L, \varphi, T_{\mathrm{s}}) \tag{2.6.17}$$

- **Caplets and Floorlets** We consider a Caplet that has a Nominal Amount $L$, a strike rate $X$, and accrual period $(t_1, t_2)$. We assume that the exercise date coincides with the start of the accrual period $t_1$, and that the interest payment is made in arrears, at the end of the accrual period $t_2$.

  The value of the Caplet $V_{\mathrm{C}}$ is the same as that of a Put Option on a ZCB, where the strike of the ZCB option is the Nominal of the Caplet and the Nominal of the ZCB Option is the Nominal with accrued interest at the Caplet Strike Rate.

$$V_{\mathrm{C}}\left(L, X, t_1, t_2\right) = V_{\mathrm{ZCBO}}(L\left(1 + X\left(t_2 - t_1\right)\right), L, t_1, t_2, -) \tag{2.6.18}$$

  Floorlets are similarly priced using Call Options on ZCBs:

$$V_{\mathrm{F}}\left(L, X, t_1, t_2\right) = V_{\mathrm{ZCBO}}(L\left(1 + X\left(t_2 - t_1\right)\right), L, t_1, t_2, +) \tag{2.6.19}$$

### 2.6.4 Swaption and Caplet pricing using the Black Model

In order to calibrate the HW model to Caps or Swaptions, we must be able to compute Caplet and/or Swaption prices from quoted Black volatilities, as the HW parameters are determined by fitting the instrument prices under the HW model.

- **Swaptions** We consider a Swaption where the underlying Swap has Strike Rate $s_K$ (*i.e.*, the rate paid on the fixed leg), Nominal $L$, Start Date $T_{\mathrm{s}}$, End date $T_{\mathrm{e}}$, and Tenor $\tau$. We take the Expiry (Exercise) date of the Swaption to be equal to the Start Date $T_{\mathrm{s}}$ of the underlying Swap. Given a quoted Black Swaption Volatility $\sigma_{\mathrm{B}}$, the Black Payer and Receiver Swaption prices are given by:[4]

$$V_S^{(\mathrm{Black})}\left(L, s_K, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, +, \sigma_{\mathrm{B}}\right) = LA\left[s_0 N\left(d_1\right) - s_K N\left(d_2\right)\right] \tag{2.6.20}$$

$$V_S^{(\mathrm{Black})}\left(L, s_K, T_{\mathrm{s}}, T_{\mathrm{e}}, \tau, -, \sigma_{\mathrm{B}}\right) = LA\left[s_K N\left(-d_2\right) - s_0 N\left(-d_1\right)\right] \tag{2.6.21}$$

  where:

$$d_1 \equiv \frac{\ln\left(\frac{s_0}{s_K}\right) + \frac{1}{2}\sigma_{\mathrm{B}}^2 T}{\sigma_{\mathrm{B}}\sqrt{T}} \tag{2.6.22}$$

$$d_2 \equiv d_1 - \sigma_{\mathrm{B}}\sqrt{T} \tag{2.6.23}$$

$$s_0 \equiv \frac{P_0(T_{\mathrm{s}}) - P_0(T_{\mathrm{e}})}{A} \tag{2.6.24}$$

[3] We note that in the case of Swaptions we refer to Payer/Receiver Swaptions rather than Calls/Puts.

[4] Here we are assuming the Act/365 Day Count Convention. For other DCCs we need to replace the $t_i - t_{i-1}$ terms with suitable Day Count Fractions $\tau(t_{i-1}, t_i)$.

$$A \equiv \sum_{i=1}^{N} (T_i - T_{i-1})\, P_0(T_i) \tag{2.6.25}$$

$s_0$ is the Swap Rate for the underlying Forward Starting Swap, namely the Fixed Rate that renders the value of the Swap zero.

- **Caplets and Floorlets** We consider a Caplet that has a Nominal Amount $L$, a strike rate $X$, and accrual period $(t_1, t_2)$. We assume that the exercise date coincides with the start of the accrual period $t_1$, and that the interest payment is made in arrears, at the end of the accrual period $t_2$. If the Black volatility of the underlying forward rate is $\sigma_{\mathrm{B}}$, then the Caplet and Floorlet prices are:[5]

$$V_{\mathrm{C}}^{(\mathrm{Black})}\left(L, X, t_1, t_2, \sigma_{\mathrm{B}}\right) = L\left(t_2 - t_1\right) P_0\left(t_2\right)\left(f(0, t_1, t_2) N\left(d_1\right) - X N\left(d_2\right)\right) \tag{2.6.26}$$

$$V_{\mathrm{F}}^{(\mathrm{Black})}\left(L, X, t_1, t_2, \sigma_{\mathrm{B}}\right) = L\left(t_2 - t_1\right) P_0\left(t_2\right)\left(X N\left(-d_2\right) - f\left(0, t_1, t_2\right) N\left(-d_1\right)\right) \tag{2.6.27}$$

where

$$d_1 = \frac{\ln\left(\frac{f(0,t_1,t_2)}{X}\right) + \frac{1}{2}\sigma_{\mathrm{B}}^2 t_1}{\sigma_{\mathrm{B}}\sqrt{t_1}} \tag{2.6.28}$$

$$d_2 = d_1 - \sigma_{\mathrm{B}}\sqrt{t_1} \tag{2.6.29}$$

### 2.6.5 Fitting Procedure

In order to fit the HW model to a collection of calibrating instruments quoted in the form of Black volatilities, the general procedure is to define a loss function as the sum of the absolute or squared differences between the Black prices of the instruments derived directly from the quoted volatilities and the prices from the HW model for a given choice of $a$ and $\sigma$. This loss function is then minimised over $a$ and $\sigma$.

- **ATM Swaption Fitting** At-the-Money (ATM) Swaption volatilities are quoted for given maturities, expiries and frequencies, giving a sequence of Payer Swaptions[6] with defining properties $T_{\mathrm{s},j}, T_{\mathrm{e},j}, \tau_j$ (which in turn generate $N_j$ cashflow details with payment times $T_{i,j}$ for $i = 1, \ldots, N_j$). The Notionals $L_j$ are arbitrary and can be taken to be 1. Because the Swaptions are ATM, the corresponding Strikes $s_{K,j}$ are defined to be the Swap Rates $s_{0,j}$ of the underlying Forward Starting Swaps to zero, *viz*:

$$s_{K,j} = s_{0,j} \tag{2.6.30}$$

$$s_{0,j} \equiv \frac{P_0(T_{\mathrm{s},j}) - P_0(T_{\mathrm{e},j})}{A_j} \tag{2.6.31}$$

$$A_j \equiv \sum_{i=1}^{N_j} (T_{i,j} - T_{i-1,j})\, P_0(T_{i,j}) \tag{2.6.32}$$

The loss function can be defined as:

$$\mathcal{L}(a, \sigma) \equiv \sum_j \left[V_{\mathrm{s}}\left(1, s_{0,j}, T_{\mathrm{s},j}, T_{\mathrm{e},j}, \tau_j, +\right) - V_S^{(\mathrm{Black})}\left(1, s_{0,j}, T_{\mathrm{s},j}, T_{\mathrm{e},j}, \tau_j, +, \sigma_{\mathrm{B},j}\right)\right]^2 \tag{2.6.33}$$

where the sum is over all of the calibrating Swaptions and $\sigma_{\mathrm{B},j}$ is the quoted Black volatility for the $j$th instrument. This function can be minimised by means of, *e.g.*, the Downhill Simplex Method [6]. Alternatively, it is common practice to fix a standard value for $a$, *e.g.*, $a = 0.03$, and then minimise the loss function over $\sigma$ alone.

[5] Here we are assuming the Act/365 Day Count Convention. For other DCCs we need to replace the $t_2 - t_1$ terms with suitable Day Count Fractions $\tau(t_1, t_2)$.

[6] Because the Swaptions are ATM, the instruments can be taken to be either Payer or Receiver Swaptions.

- **ATM Cap Fitting** Cap volatilities are usually quoted as a flat Black volatility that, when applied to all of the Caplets that comprise the Cap, gives the same Cap price as the market would give for the sum of the prices of the individual Caplets (which would each in general have different implied volatilities. In the case of ATM Caps the Caplet Strikes are taken to be the forward rates corresponding to the Caplet accrual periods. In order to fit the HW model to a single ATM Cap with Maturity $T$ and Tenor $\tau$ (usually three months), we generate a set of $N$ Caplets [7] that comprise the Cap. The Caplets have arbitrary Notionals $L_j$ that we can take to be 1, accrual period Start Dates (also the Expiry Dates) $t_{1,j} \approx j\tau$, accrual period End Dates $t_{2,j} \approx (j+1)\tau$, and ATM Strikes, *viz*:
$$X_j \equiv f(0, t_{1,j}, t_{2,j}) \tag{2.6.34}$$
With the quoted Black flat volatility $\sigma_{\mathrm{B}}$, we can define a loss function for the Cap as:
$$\mathcal{L}(a, \sigma) \equiv \sum_{j=1}^{N} \left[ V_{\mathrm{C}}\left(1, X_j, t_{1,j}, t_{2,j}\right) - V_{\mathrm{C}}^{(\mathrm{Black})}\left(1, X_j, t_{1,j}, t_{2,j}, \sigma_{\mathrm{B}}\right) \right]^2 \tag{2.6.35}$$
This loss function is the sum of the squared distances between the Black and HW Caplet prices. An alternative loss function is the distance between the Black and HW prices for the Cap as a whole, *viz*:
$$\mathcal{L}(a, \sigma) \equiv \left[ \sum_{j=1}^{N} V_{\mathrm{C}}\left(1, X_j, t_{1,j}, t_{2,j}\right) - \sum_{j=1}^{N} V_{\mathrm{C}}^{(\mathrm{Black})}\left(1, X_j, t_{1,j}, t_{2,j}, \sigma_{\mathrm{B}}\right) \right]^2 \tag{2.6.36}$$
As per the Swaptions case, these loss functions can minimised to obtain $a$ and $\sigma$, or we can fix $a = 0.03$ and then just optimise over $\sigma$.

- **More General Fitting** The above loss functions for Swaptions and single Caps can be extended straightforwardly to perform fits to a strip of Caps, or a combination of Swaptions and Caps. Clearly, also, it is possible to fit to individual Caplets, or combinations of Caplets and Swaptions, if individual Caplet volatilities are available.

## 2.7 HWAUTO Process for Risk Neutral Zero Curve Evolution: Implementation Details and Calibration

The simulation process for the yield curve under the HWAUTO stochastic process is carried out in the following sequence of steps:

1. We assume that we have evolved the term structure up to $t = t_0$ from its initial state through the evolution on a given simulation path and we wish to generate the term structure at a later time $t > t_0$. In doing so we will have evolved the short rate on the path up to $t = t_0$ and hence we know $r(t_0)$.

2. We use (2.5.18) to evolve the short rate to time $t$, *i.e.*, we calculate $r(t)$. The mean $\mu(t_0, t)$ is calculated from (2.5.26) with the $F_0(t_0)$ and $F_0(t)$ calculated from (2.2.4) which in turn requires the compuation of the initial term structure and its first derivative. (It is not necessary to compute $\theta(t)$ and hence the second derivative $R_0''(t)$ of the term structure in order to compute the distribution function for $r(t)$.) The first derivative $R_0'(t)$ is currently computed using the simple finite difference (2.3.2). The s.d. $\Sigma(t_0, t)$ is computed from (2.5.20). A "covariated" normal deviate is parsed to the curve evolution function (a component of the $dz$ vector). Formally it can be viewed as taking the form $\sigma\phi$. We must divide out the $\sigma$ component so that we are just left with the correlated deviate $\phi$, which is then combined with $\mu(t_0, t)$ and $\Sigma(t_0, t)$ in (2.5.18) to define $r(t)$. Rather than first dividing $\sigma$ out of the covariated deviate to obtain $\phi$ and multiplying by $\Sigma(t_0, t)$, we actually compute the function $\Sigma(t_0, t)/\sigma$ and then multiply the result by the covariated deviate instead.

3. The normal deviate $\phi$ must be correlated with the normal deviates used to evolve other curves (zero curves that are also evolved using the HW model, or otherwise) in the simulation. This is achieved by proxying short rate correlations with correlations with the shortest dated rate on the curve of interest. In fact, the deviates used in the stochastic process evolution methods are "covariated" in that we start with a vector of uncorrelated

[7] Because the instruments are ATM, they can be taken to be either Caps or Floors. The number of Caplets $N$ is assumed to be defined such that $(N+1)\tau = T$. The accrual periods are approximately (*i.e.*, up to business day adjustments) $(\tau, 2\tau)$, $(2\tau,\ 3\tau), \ldots, (N\tau, (N+1)\tau = T)$. Hence there is no initial Caplet with accrual period $(0, \tau)$ as this Caplet would be subject to immediate exercise and, being ATM, would be worth zero. The maturity date $T$ is the end date of the accrual period of the final Caplet, not the exercise date of the final Caplet, which is approximately $N\tau = T - \tau$. We assume that for each Caplet the accruals are set in advance and paid in arrears and that there is no lag between the fixing (exercise) date and the start of the accrual period nor between the end of the accrual period and payment. We also assume that the end date of the $j$th Caplet is the start date of the $(j+1)$th Caplet, *i.e.*, $t_{1,j+1} = t_{2,j}$ for $j = 1, \ldots, N-1$.

normal deviates corresponding to all of the curve factors (as opposed to all of the curves' rate points) and then multiply the by factor covariance matrix to give a vector of covariated deviates: one for each rate point on each curve. We denote the covariated normal deviates vector by $dz$. In the case of the HW short rate $r(t)$, which will be identified with the shortest dated rate point on the zero curve, the rate evolution function will be parsed a $dz$ component which will be identified with $\sigma\phi$, and so to extract the $\phi$ we divide out the $\sigma$.

4. We next compute all of the evolved zero rates at time $t$ from $r(t)$. We loop over all of the term points $t_1, \ldots, t_n$ on the zero curve and compute $R(t, t+t_i)$ using (2.4.6) with $T = t + t_i$, *viz*:

$$R(t, t+t_i) = \frac{1}{t_i}\left[-\ln A(t, t+t_i) + B(t, t+t_i)r(t)\right] \tag{2.7.1}$$

   To do so we need to evaluate the quantities $B(t, t+t_i)$ and $A(t, t+t_i)$ using (2.4.2) and (2.4.5). These functions require the valuation of $P_0(t+t_i)$, $P_0(t)$ and $F_0(t)$ which in turn depend on the initial term structure and its first derivative through (2.2.1) and (2.2.4). Of course, $F_0(t)$ has already been computed in the calculation of $r(t)$.

5. In the case of the first iteration, when we evolve from $t_0 = 0$ out to the first MC credit node, we take $r(t_0) = R_0(t_1)$ where $t_1$ is the shortest dated term on the yield curve. As discussed, we assume here that flat extrapolation is used outside of the yield curve's prescribed terms.

### 2.7.1 Calibration

For the HWAUTO process, the reversion speed parameter $a$ is supplied exogenously. For each zero curve that uses the HWAUTO process for risk neutral evolution, a em ReversionFactor value is loaded by the user. Note that, even though the Vector Risk system allows a term structure for process parameters, the HW process only requires this parameter for the instantaneous short rate, so it is only necessary to supply a RevesionFactor parameter value for one (nominal) term (*e.g.*, 1). If a term structure is supplied, the shortest dated value will be used.

With $s$ set, the short rate volatility $\sigma$ is computed by fitting the HW model to a strip of Black Cap volatilities by minimising the loss function (2.6.35). The Cap volatilities are taken from the Cap volatility curve of the same currency. *E.g.*, if we are calibrating $\sigma$ for the MM-ZERO-SWAP-GBP or MM-ZERO-OIS-GBP, we obtain the strip of Black Cap volatilities from the MM-CAPVOL-CAP-GBP curve. Each point on this curve is identified with a Cap calibration instrument for the purpose of Eq, (2.6.35), and all instruments are weighted equally in the loss function.

Vector Risk Analytics can perform HW calibrations to Swaptions or a mixture of Swaptions and Caps, in the context of exotic interest rate derivative pricing. It is expected that HW processes for risk neutral zero curve evolution will be introduced with these extra calibration options.