American Option

Vector Risk Pty Ltd

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1 Inputs to Function

Description	*Symbol*	*min*	*max*	*Reasonable range*
Underlying	S	0^+	$+\infty$	
Strike	X	0^+	$+\infty$	
Continuous risk-free interest rate	r	0^+	$+\infty$	
Continuous secondary rate	q	0^+	$+\infty$	
Volatility	σ	0^+	$+\infty$	
Time to maturity	T	0^+	$+\infty$	
Put or Call	*indicator*	–	–	"P" or "C"

Table 1: Inputs for American Option pricing function

2 Formula

Bjerksund and Stensland (2002) approximation gives the lower bound of the value, f_{call}, of an American *call* option as

$$\boxed{\begin{aligned} f_{call} = {} & \alpha(I_2)\,S^\beta - \alpha(I_2)\,\phi(S,t_1\,|\,\beta,I_2,I_2) + \phi(S,t_1\,|\,1,I_2,I_2) - \phi(S,t_1\,|\,1,I_1,I_2) \\ & - X\phi(S,t_1\,|\,0,I_2,I_2) + X\phi(S,t_1\,|\,0,I_1,I_2) + \alpha(I_1)\,\phi(S,t_1\,|\,\beta,I_1,I_2) - \alpha(I_1)\,\Psi(S,T\,|\,\beta,I_1,I_2,I_1,t_1) \\ & + \Psi(S,T\,|\,1,I_1,I_2,I_1,t_1) - \Psi(S,T\,|\,1,X,I_2,I_1,t_1) - X\Psi(S,T\,|\,0,I_1,I_2,I_1,t_1) + X\Psi(S,T\,|\,0,X,I_2,I_1,t_1)\,, \end{aligned}}$$

where

$$\alpha(I) = (I-X)I^{-\beta} \qquad \beta = \left(\frac{1}{2} - \frac{r-q}{\sigma^2}\right) + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}},$$

and the trigger prices are defined as

$$I_1 = B_0 + (B_\infty - B_0)\left(1 - e^{h(t_1)}\right) \qquad I_2 = B_0 + (B_\infty - B_0)\left(1 - e^{h(T)}\right)$$

$$h(t) = -\left[(r-q)t + 2\sigma\sqrt{t}\right]\frac{X^2}{(B_\infty - B_0)\,B_0} \qquad t_1 = \frac{1}{2}\left(\sqrt{5}-1\right)T$$

$$B_\infty = \frac{\beta}{\beta-1}X \qquad B_0 = \max\left(X, \frac{r}{q}X\right),$$

and the function $\phi(S,t_1\,|\,\gamma,H,I_2)$ is given by

$$\phi(S,t_1\,|\,\gamma,H,I_2) = e^{\lambda t_1}S^\gamma\left[N(-d_1) - \left(\frac{I_2}{S}\right)^\kappa N(-d_2)\right],$$

where

$$d_1 = \frac{\ln \frac{S}{H} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] t_1}{\sigma\sqrt{t_1}} \qquad d_2 = \frac{\ln \frac{I_2^2}{SH} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] t_1}{\sigma\sqrt{t_1}}$$

$$\lambda = (\gamma - 1) r - \gamma q + \frac{1}{2}\gamma(\gamma - 1)\sigma^2 \qquad \kappa = \frac{2(r-q)}{\sigma^2} + (2\gamma - 1),$$

and the function $\Psi(S, T \mid \gamma, H, I_2, I_1, t_1)$ is given by

$$\begin{aligned}
&\Psi(S, T \mid \gamma, H, I_2, I_1, t_1) \\
&= e^{\lambda T} S^{\gamma} \left[N_2(-e_1, -f_1; \rho) - \left(\frac{I_2}{S}\right)^{\kappa} N_2(-e_2, -f_2; \rho) - \left(\frac{I_1}{S}\right)^{\kappa} N_2(-e_3, -f_3; -\rho) + \left(\frac{I_1}{I_2}\right)^{\kappa} N_2(-e_4, -f_4; -\rho) \right],
\end{aligned}$$

where

$$e_1 = \frac{\ln \frac{S}{I_1} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] t_1}{\sigma\sqrt{t_1}} \qquad e_2 = \frac{\ln \frac{I_2^2}{SI_1} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] t_1}{\sigma\sqrt{t_1}}$$

$$e_3 = \frac{\ln \frac{S}{I_1} - \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] t_1}{\sigma\sqrt{t_1}} \qquad e_4 = \frac{\ln \frac{I_2^2}{SI_1} - \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] t_1}{\sigma\sqrt{t_1}}$$

$$f_1 = \frac{\ln \frac{S}{H} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] T}{\sigma\sqrt{T}} \qquad f_2 = \frac{\ln \frac{I_2^2}{SH} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] T}{\sigma\sqrt{T}}$$

$$f_3 = \frac{\ln \frac{I_1^2}{SH} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] T}{\sigma\sqrt{T}} \qquad f_4 = \frac{\ln \frac{SI_1^2}{HI_2^2} + \left[r - q + \left(\gamma - \frac{1}{2}\right)\sigma^2\right] T}{\sigma\sqrt{T}}$$

$$\rho = \sqrt{\frac{t_1}{T}}.$$

B_T is defined as the optimal exercise price for an American call option, with exercise price X and time to maturity T, where no restriction applies to the shape of the exercise boundary.

The value, f_{put}, of the corresponding *put* option is approximated[1] by

$$P(S, X, r, q, \sigma, T) = C(X, S, q, r, \sigma, T),$$

where $P()$ and $C()$ are the valuation functions for American put and call options respectively.

3 Properties of Instrument

An American call (put) option is the right, but not the obligation, to buy (sell) the underlying asset S for the given exercise price X. Unlike an European option, where exercise is possible only at maturity T, an American option can be exercised at any t such that $0 \le t \le T$.

At any time t, where $0 \le t \le T$, the holder of an American option would exercise the option if and only if it is optimal to do so, i.e. the immediate payoff of the option is higher than the value of keep on holding the option. If we denote the exercise time as τ, with value the underlying asset as S_τ, the immediate payoff by exercising the American option is illustrated in Table 2.

Option Type	Payoff
Call	$S_\tau - X$
Put	$X - S_\tau$

Table 2: Payoff at time of exercising for American option

[1] See Bjerksund and Stensland (2002) and Haug (2007) for details.

Bibliography

Petter Bjerksund and Gunnar Stensland. Closed-form valuation of american options. Discussion paper 9, Norwegian School of Economics and Business Administration, Department of Finance and Management Science, 2002.

Espen Gaarder Haug. *The Complete Guide To Option Pricing Formulas*. McGraw Hill, New York, 2nd edition, 2007.