Arithmetic Average Rate Option

Vector Risk Pty Ltd

August 13, 2013

Version 8.0.2482

1 Formula

Curran (1994) describes two methods to approximate the value of an arithmetic average rate call option. We use a similar approach[1] to give the lower bound of the value of a *forward starting* arithmetic average rate option as

$$\mathrm{ARO}(\phi, \mathbf{A}, X) = \phi e^{-r_T T} \sum_{i=1}^{n} w_i \mathbb{E}\left(S_{t_i}\right) N\left(\phi \frac{\mathrm{Cov}\left(\ln S_{t_i}, Y\right) + \mathbb{E}(Y) - Y_X^*}{\sqrt{\mathrm{Var}(Y)}}\right) - \phi X e^{-r_T T} N\left(\phi \frac{\mathbb{E}(Y) - Y_X^*}{\sqrt{\mathrm{Var}(Y)}}\right),$$

where $\mathbf{A} = \{(w_1, S_{t_1}), \ldots, (w_n, S_{t_n})\}$ is the set of scaled weights and underlying values at each averaging date t_i, and Y_X^* is found by solving[2]

$$\sum_{i=1}^{n} w_i \mathbb{E}\left(S_{t_i}\right) \exp\left\{\frac{\mathrm{Cov}\left(\ln S_{t_i}, Y\right)}{\mathrm{Var}(Y)}\left[Y_X^* - \mathbb{E}(Y)\right] - \frac{1}{2}\frac{\left[\mathrm{Cov}\left(\ln S_{t_i}, Y\right)\right]^2}{\mathrm{Var}(Y)}\right\} = X$$

with

$$\mathbb{E}\left(S_{t_i}\right) = S e^{\left(r_{t_i} - q_{t_i}\right) t_i} \qquad \mathbb{E}(Y) = \sum_{i=1}^{n} w_i \left[\ln \mathbb{E}\left(S_{t_i}\right) - \frac{1}{2}\sigma_{t_i}^2 t_i\right] \qquad \mathrm{Var}(Y) = \sum_{i=1}^{n} w_i \sigma_{t_i}^2 t_i \left(w_i + 2\sum_{j=1}^{n} w_j - 2\sum_{j=1}^{i} w_j\right)$$

$$\mathrm{Cov}\left(\ln S_{t_1}, Y\right) = \sigma_{t_1}^2 t_1 \qquad \mathrm{Cov}\left(\ln S_{t_i}, Y\right) = \mathrm{Cov}\left(\ln S_{t_{i-1}}, Y\right) + \left(\sigma_{t_i}^2 t_i - \sigma_{t_{i-1}}^2 t_{i-1}\right)\left(\sum_{j=1}^{n} w_j - \sum_{j=1}^{i-1} w_j\right) \text{ (for } i > 1\text{).}$$

ϕ	Option Type
-1	Put
1	Call

The value of an *in-progress* arithmetic average rate option where the underlying has already been observed at times t_i $(1 \le i \le m)$ can be represented as a scaled forward starting arithmetic average rate option with adjusted 'active' averaging objects and strike, i.e.

$$\mathrm{ARO}\left(\phi, \widetilde{\mathbf{A}}, \widetilde{X}\right),$$

where

$$\widetilde{\mathbf{A}} = \left\{\left(w_{m+1}, S_{t_{m+1}}\right), \ldots, \left(w_n, S_{t_n}\right)\right\} \qquad \widetilde{X} = X - A_H,$$

and

$$A_H = \sum_{i=1}^{m} w_i S_i.$$

When $\widetilde{X} \le 0$, the value of an in-progress arithmetic average rate call option is

$$e^{-r_T T}\left[\sum_{i=m+1}^{n} w_i \mathbb{E}\left(S_i\right) + A_H - X\right],$$

and the value of an in-progress arithmetic average rate put option is 0.

[1] The naive method as described in Curran (1994)

[2] Any root finding method is acceptable

2 Properties of Instrument

The payoff for an arithmetic average rate call option at expiry is given by

$$C_T = (A_T - X)^+ \,, \tag{1}$$

where the weighted arithmetic mean of the underlying at n pre-defined observation times t_i is given by

$$A_T = \sum_{i=1}^{n} w_i S_{t_i} \,, \tag{2}$$

where $(w_i,\, S_{t_i})$ is the (weight, underlying) pair at t_i.

Similarly, the payoff for an arithmetic average rate put option at expiry is given by

$$P_T = (X - A_T)^+ \,. \tag{3}$$

3 Numerical Analysis

Given the Curran naive method provides a lower bound on the price, we perform 100 million paths Monte Carlo simulation to analyse the accuracy of the method. Rather than just analyse the Curran naive method, we also analyse the Curran sophisticated method [3], which is also available in our Risk Engine.

We use 100 million paths simulation as we want to have both Curran methods give prices less than the Monte Carlo simulation method. Unfortunately, as we will see later on, there are a few cases where this objective is not achieved.

Given c_2 has an analytical solution, we could possibly reduce the standard error of our Monte Carlo simulation by simulating the value of c_1 with importance sampling and add the analytical solution of c_2 to get the price of the average rate options. However, we chose not to use this method as we do not want to possibly introduce an error in the analysis if the importance sampling is not implemented properly.

We only consider the forward starting call options. In progress options can be transformed into forward starting option as described in Section 1. While put options can be valued using the put-call parity.

3.1 Numerical Results

Firstly, we look at the effect of time till maturity. We construct forward starting average rate call options with

- the spot price of \$100,
- the strike price of \$90,
- the zero rate as 5%,
- the dividend yield as 3%, and
- the volatility as 30%.

The time till maturity of the options varies from 1 month to 10 years, with increment of 1 month.

Figure 1 shows that the error increases as time till maturity increases, with the error of the sophisticated method increases at a slower rate. We note that the error is still acceptable for both methods.

Next, we look at the effect of strike price. We construct three sets of forward starting average rate call options with

- the spot price of \$100,
- the zero rate as 5%,
- the dividend yield as 3%, and
- the volatility as 30%.

[3] Curran (1994) estimated c_1 by imposing a mesh on the geometric mean and performed numerical integration on it. He called this method as 'sophisticated'.

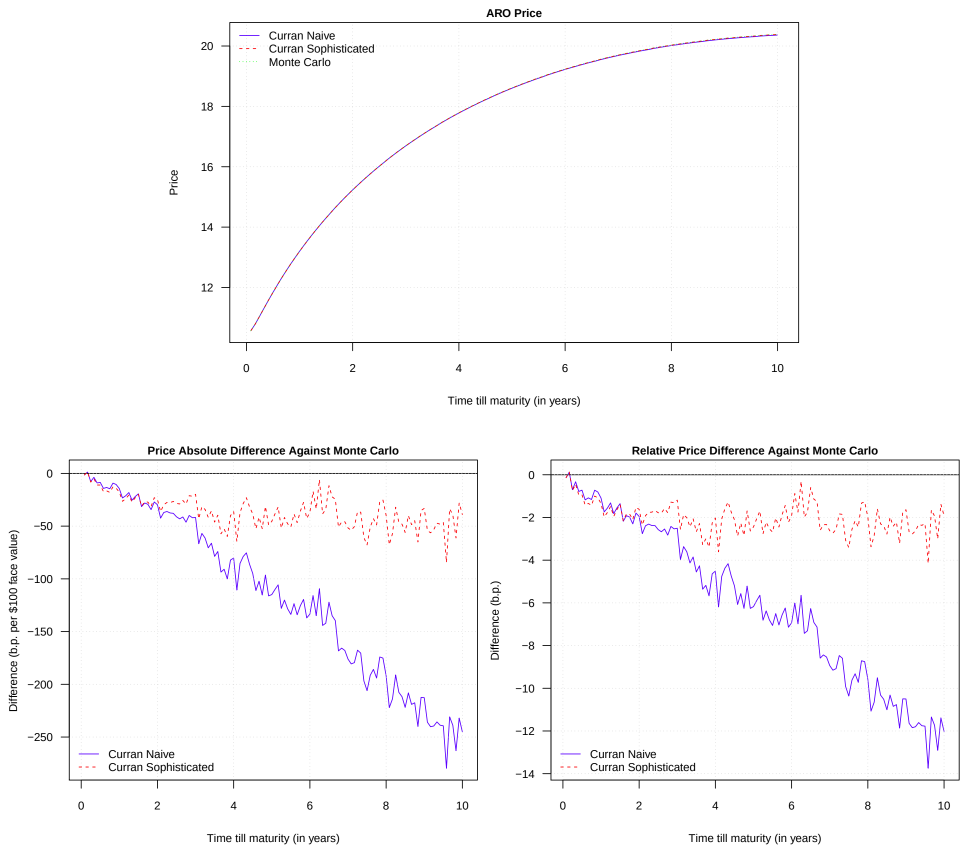

Figure 1: Forward starting ARO with $S = 100$, $X = 90$, $r = 0.05$, $q = 0.03$ and $\sigma = 0.30$

The time till maturity for the three sets of options are 1 year, 5 years and 10 years respectively. The strike price of the options varies from 50 to 150, with increment of 1.

Figure 2, Figure 3 and Figure 4 show that there is a trend that error increases as the strike price increases. Also, the sophisticated method performs significantly better than the naive method, though with a relative error of less than 0.4% for most cases, the naive method is still acceptable. It is interesting to note that for $T = 1$, the Monte Carlo method gives prices marginally less than both Curran methods when the option is far in the money. We can avoid this problem by increasing the number of paths in our simluation, though we believe it is unncessary given the small relative error that we observed for these scenarios.

Finally, we look at the effect of volatility. We construct three sets of forward starting average rate call options with

- the spot price of $100,
- the strike price of $90,
- the zero rate as 5%, and
- the dividend yield as 3%.

The time till maturity for the three sets of options are 1 year, 5 years and 10 years respectively. The volatility of the options varies from 1% to 100%, with increment of 1%.

Figure 5, Figure 6 and Figure 7 show that error increases as the volatility increases. Also, the sophisticated method performs significantly better than the naive method, especially for the cases of high volatility. In the cases of high volatility, the sophisticated method gives a very low relatively error at the cost of more computation. It is interesting to note that in the cases of $T = 10$ and volatility greater than 80%, the sophisticated method gives a higher price than the Monte Carlo method. Despite the naive method struggles in the cases of high volatility, its performance is acceptable for normal market condition.

Based on the analysis, we conclude that the lower bound price given by Curran naive method is very close to the true option price in normal market condition; while the lower bound price given by Curran sophisticated method performs better, especially in unusual market condition.

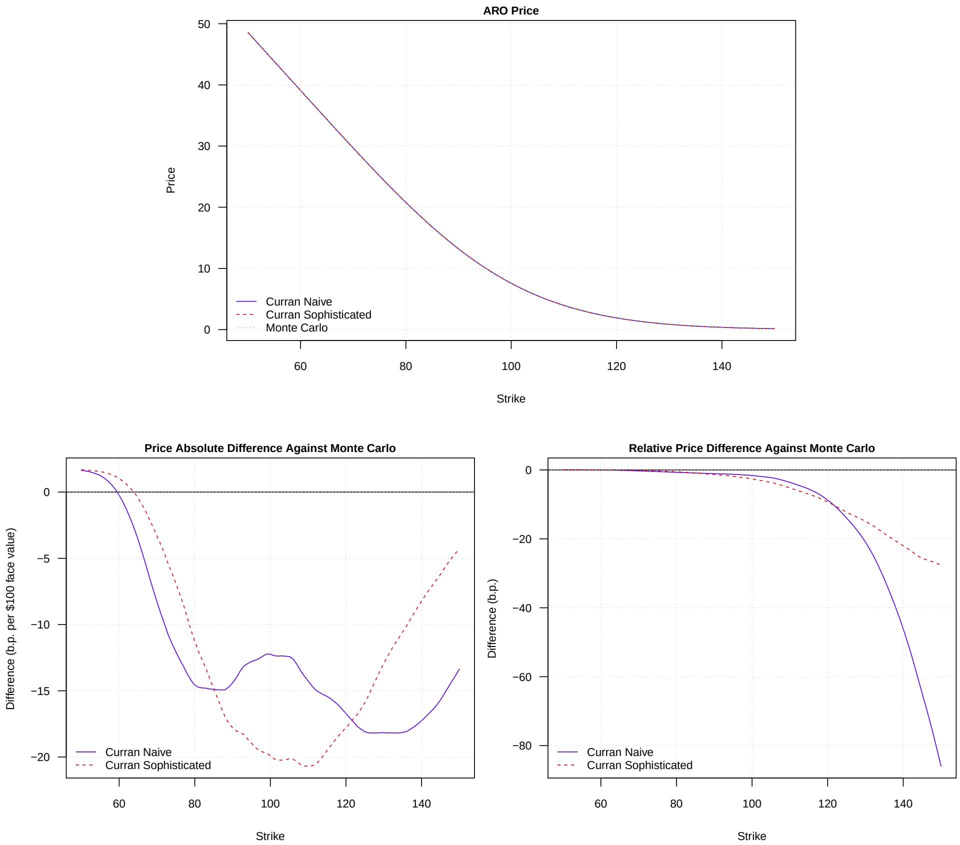

Figure 2: Forward starting ARO with $S = 100$, $r = 0.05$, $q = 0.03$, $\sigma = 0.30$ and $T = 1$

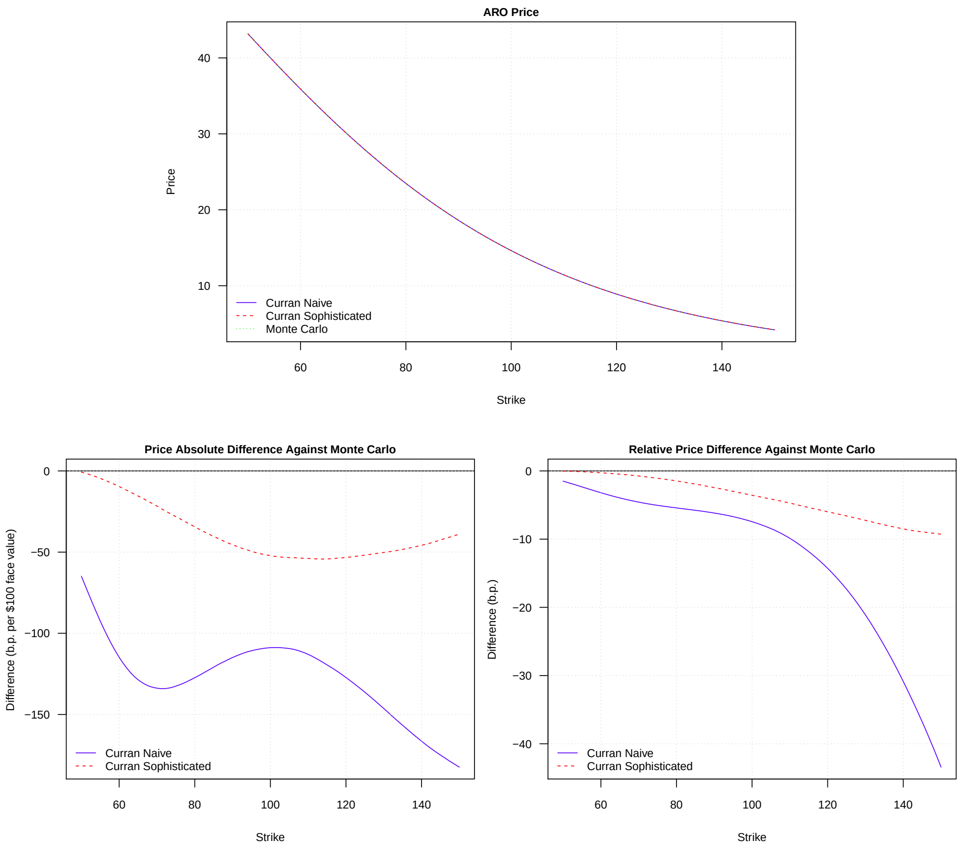

Figure 3: Forward starting ARO with $S = 100$, $r = 0.05$, $q = 0.03$, $\sigma = 0.30$ and $T = 5$

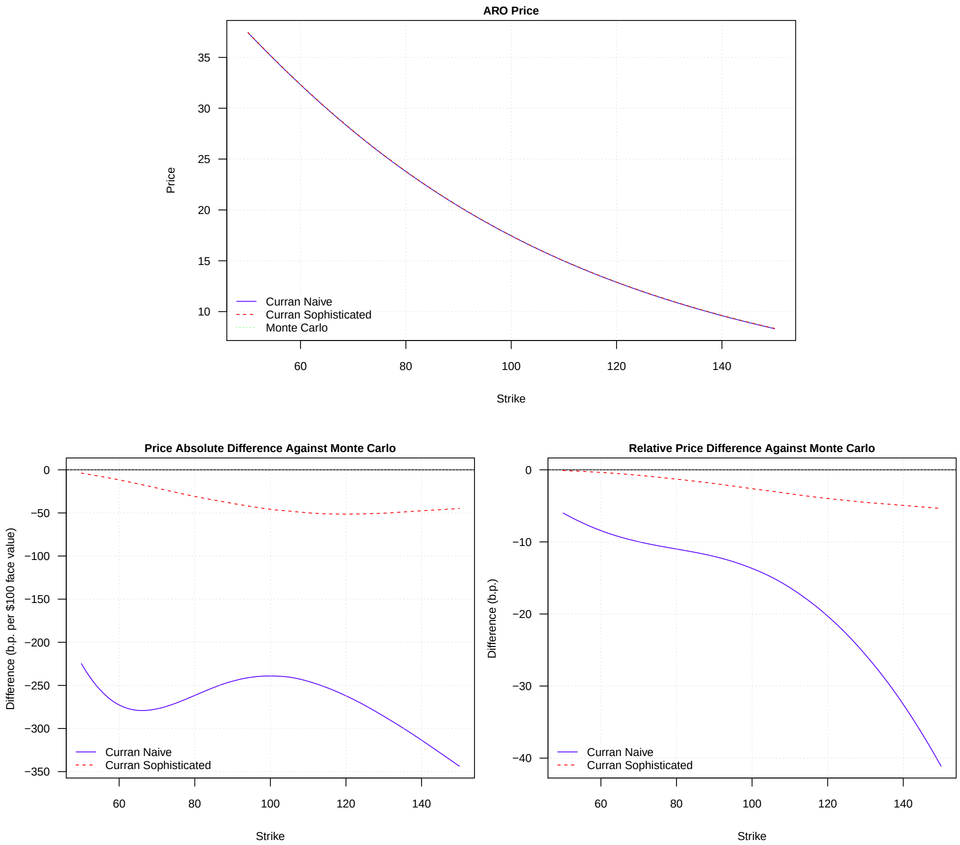

Figure 4: Forward starting ARO with $S = 100$, $r = 0.05$, $q = 0.03$, $\sigma = 0.30$ and $T = 10$

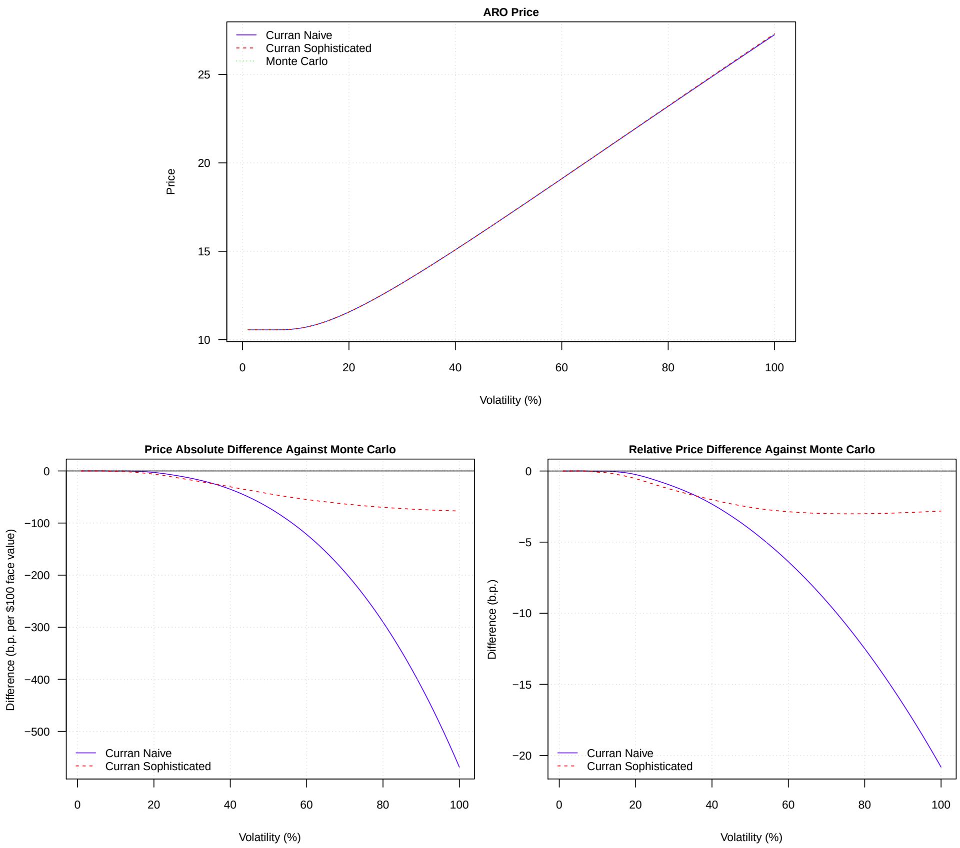

Figure 5: Forward starting ARO with $S = 100$, $X = 90$, $r = 0.05$, $q = 0.03$ and $T = 1$

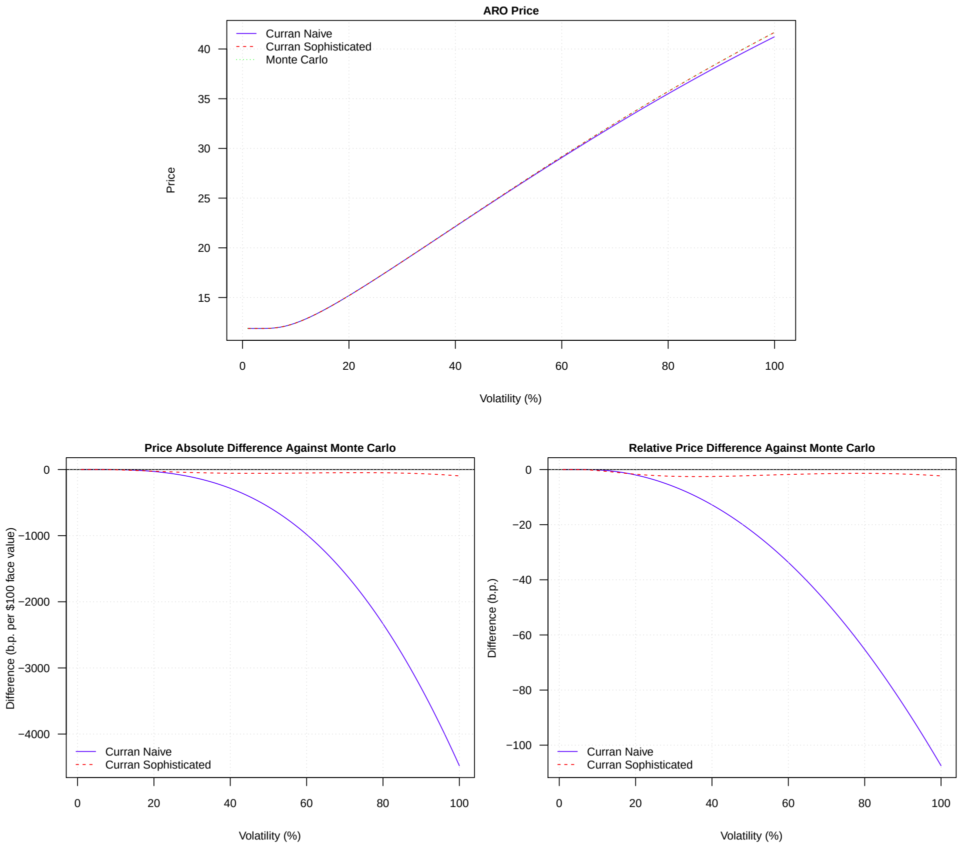

Figure 6: Forward starting ARO with $S = 100$, $X = 90$, $r = 0.05$, $q = 0.03$ and $T = 5$

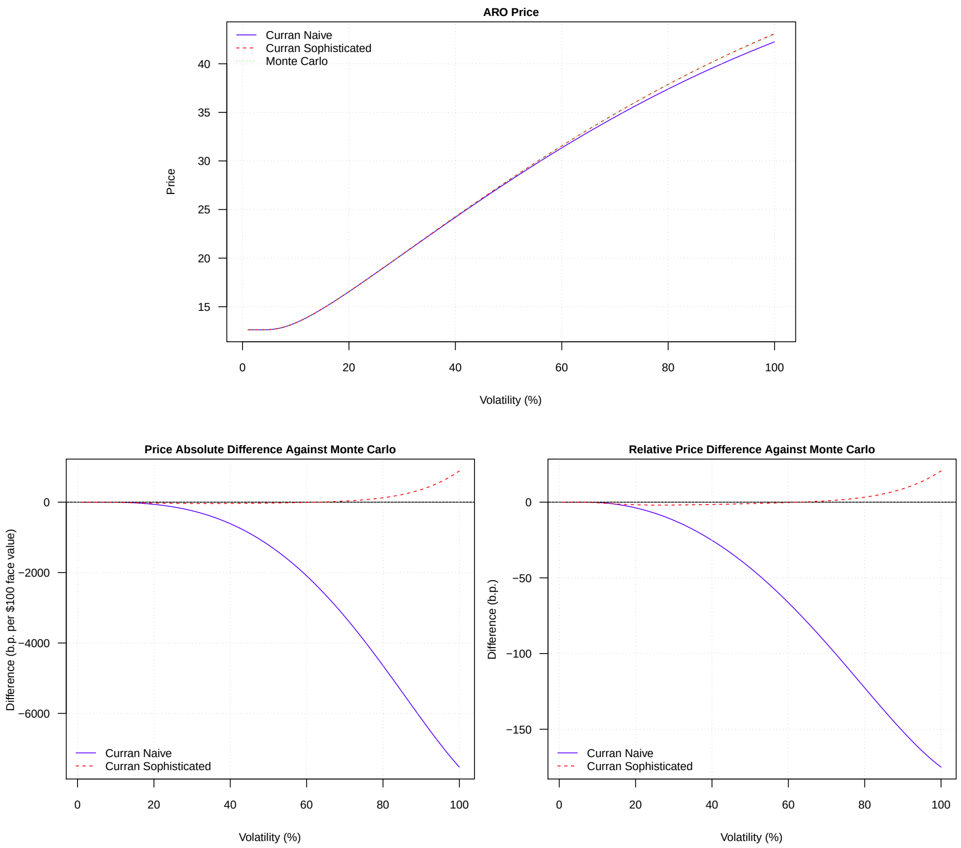

Figure 7: Forward starting ARO with $S = 100$, $X = 90$, $r = 0.05$, $q = 0.03$ and $T = 10$

Bibliography

Michael Curran. Valuing asian and portfolio options by conditioning on the geometric mean price. *Management Science*, 40(12):1705–1711, 1994.

Glossary

Risk Engine The Vector Risk market risk and credit risk system.