Single Partial Barrier Late Start Option

Vector Risk Pty Ltd

April 13, 2017

Version 8.0.7970

1 Input to Function

Description	*Symbol*	*min*	*max*	*Reasonable range*
Underlying price	S	0^+	$+\infty$	
Strike price	X	0^+	$+\infty$	
Barrier level	H	0^+	$+\infty$	
Continuous risk-free interest rate until t_1	r_1	0^+	$+\infty$	
Continuous secondary rate until t_1	q_1	0^+	$+\infty$	
Volatility until t_1	σ_1	0^+	$+\infty$	
Time to barrier start	t_1	0^+	$< T_2$	
Continuous risk-free interest rate until T_2	r_2	0^+	$+\infty$	
Continuous secondary rate until T_2	q_2	0^+	$+\infty$	
Volatility until T_2	σ_2	0^+	$+\infty$	
Time to option maturity	T_2	$> t_1$	$+\infty$	
Put or Call		–	–	"P", "C"
Up or Down	*indicator*	–	–	"U", "D"
In or Out		–	–	"I", "O"

Table 1: Inputs for Single Partial Barrier Late Start Option pricing function

2 Formula

The value of a *single partial barrier late start* is dependent on the option type, barrier direction, and the position of the strike with respect to the barrier, as illustrated below.

1) Down-and-in call
   Payoff: $\max(S_T - X, 0)$ if $S_t \leq H$ for some $t_1 \leq t \leq T$.
   Value: $(X > H)$: $A + C$ $\quad \eta = 1, \quad \phi = 1 \quad \xi = -1$
   Value: $(X < H)$: $E - B + D$ $\quad \eta = 1, \quad \phi = 1 \quad \xi = -1$

2) Up-and-in call
   Payoff: $\max(S_T - X, 0)$ if $S_t \geq H$ for some $t_1 \leq t \leq T$.
   Value: $(X > H)$: E $\quad \eta = -1, \quad \phi = 1 \quad \xi = -1$
   Value: $(X < H)$: $A + B - C + D$ $\quad \eta = -1, \quad \phi = 1 \quad \xi = -1$

3) Down-and-in put
   Payoff: $\max(X - S_T, 0)$ if $S_t \leq H$ for some $t_1 \leq t \leq T$.
   Value: $(X > H)$: $A + B - C + D$ $\quad \eta = 1, \quad \phi = -1 \quad \xi = -1$
   Value: $(X < H)$: E $\quad \eta = 1, \quad \phi = -1 \quad \xi = -1$

4) Up-and-in put
   Payoff: $\max(X - S_T, 0)$ if $S_t \geq H$ for some $t_1 \leq t \leq T$.
   Value: $(X > H)$: $E - B + D$ $\quad \eta = -1, \quad \phi = -1 \quad \xi = -1$
   Value: $(X < H)$: $A + C$ $\quad \eta = -1, \quad \phi = -1 \quad \xi = -1$

5) Down-and-out call
   Payoff: $\max(S_T - X, 0)$ if $S_t > H$ for all $t_1 \leq t \leq T$.
   Value: $(X > H)$: $A - C$ $\quad \eta = 1, \quad \phi = 1 \quad \xi = 1$
   Value: $(X < H)$: $B - D$ $\quad \eta = 1, \quad \phi = 1 \quad \xi = 1$

6) Up-and-out call
   Payoff: $\max(S_T - X, 0)$ if $S_t < H$ for all $t_1 \leq t \leq T$.
   Value: $(X > H)$: 0 $\quad \eta = -1, \quad \phi = 1 \quad \xi = 1$
   Value: $(X < H)$: $A - B + C - D$ $\quad \eta = -1, \quad \phi = 1 \quad \xi = 1$

7) Down-and-out put
   Payoff: $\max(X - S_T, 0)$ if $S_t > H$ for all $t_1 \leq t \leq T$.
   Value: $(X > H)$: $A - B + C - D$ $\quad \eta = 1, \quad \phi = -1 \quad \xi = 1$
   Value: $(X < H)$: 0 $\quad \eta = 1, \quad \phi = -1 \quad \xi = 1$

8) Up-and-out put
   Payoff: $\max(X - S_T, 0)$ if $S_t > H$ for all $t_1 \leq t \leq T$.
   Value: $(X > H)$: $B - D$ $\quad \eta = -1, \quad \phi = -1 \quad \xi = 1$
   Value: $(X < H)$: $A - C$ $\quad \eta = -1, \quad \phi = -1 \quad \xi = 1$

where

$$
\begin{aligned}
A &= \phi S e^{-q_2 T_2} N_2\left(\xi\eta a_1, \phi b_1; \xi\eta\phi\rho\right) - \phi X e^{-r_2 T_2} N_2\left(\xi\eta a_2, \phi b_2; \xi\eta\phi\rho\right)\\
B &= \phi S e^{-q_2 T_2} N_2\left(\eta a_1, \phi c_1; \eta\phi\rho\right) - \phi X e^{-r_2 T_2} N_2\left(\eta a_2, \phi c_2; \eta\phi\rho\right)\\
C &= \phi S e^{-q_2 T_2 + 2(\mu_2+1)\alpha\sigma_1^2 t_1} \left(\frac{H}{S}\right)^{2(\mu_2+1)} N_2\left(-\eta d_1, \eta e_1; -\rho\right) - \phi X e^{-r_2 T_2 + 2\mu_2\alpha\sigma_1^2 t_1} \left(\frac{H}{S}\right)^{2\mu_2} N_2\left(-\eta d_2, \eta e_2; -\rho\right)\\
D &= \phi S e^{-q_2 T_2 + 2(\mu_2+1)\alpha\sigma_1^2 t_1} \left(\frac{H}{S}\right)^{2(\mu_2+1)} N_2\left(-\eta d_1, \eta f_1; -\rho\right) - \phi X e^{-r_2 T_2 + 2\mu_2\alpha\sigma_1^2 t_1} \left(\frac{H}{S}\right)^{2\mu_2} N_2\left(-\eta d_2, \eta f_2; -\rho\right)\\
E &= \phi S e^{-q_2 T_2} N\left(\phi b_1\right) - \phi X e^{-r_2 T_2} N\left(\phi b_2\right),
\end{aligned}
$$

with

$$
\begin{aligned}
a_1 &= \frac{\ln\frac{S}{H} + \left(r_1 - q_1 + \frac{\sigma_1^2}{2}\right) t_1}{\sigma_1\sqrt{t_1}} & a_2 &= a_1 - \sigma_1\sqrt{t_1}\\
b_1 &= \frac{\ln\frac{S}{X} + \left(r_2 - q_2 + \frac{\sigma_2^2}{2}\right) T_2}{\sigma_2\sqrt{T_2}} & b_2 &= b_1 - \sigma_2\sqrt{T_2}\\
c_1 &= \frac{\ln\frac{S}{H} + \left(r_2 - q_2 + \frac{\sigma_2^2}{2}\right) T_2}{\sigma_2\sqrt{T_2}} & c_2 &= c_1 - \sigma_2\sqrt{T_2}\\
d_1 &= -a_1 + 2\left(\mu_2 + 1\right)\sigma_1\sqrt{t_1} & d_2 &= d_1 - \sigma_1\sqrt{t_1}\\
e_1 &= \frac{\ln\frac{H^2}{SX} - \left(r_2 - q_2 + \frac{\sigma_2^2}{2}\right) T_2 + \beta}{\sigma_2\sqrt{T_2}} & e_2 &= e_1 - \sigma_2\sqrt{T_2}\\
f_1 &= \frac{\ln\frac{H}{S} - \left(r_2 - q_2 + \frac{\sigma_2^2}{2}\right) T_2 + \beta}{\sigma_2\sqrt{T_2}} & f_2 &= f_1 - \sigma_2\sqrt{T_2}\\
\rho &= \frac{\sigma_1\sqrt{t_1}}{\sigma_2\sqrt{T_2}} & \beta &= 2\left(\mu_2 + 1\right)\sigma_2^2 T_2\\
\mu_1 &= \frac{r_1 - q_1 - \frac{\sigma_1^2}{2}}{\sigma_1^2} & \mu_2 &= \frac{\left(r_2 - q_2 - \frac{\sigma_2^2}{2}\right) T_2 - \left(r_1 - q_1 - \frac{\sigma_1^2}{2}\right) t_1}{\sigma_2^2 T_2 - \sigma_1^2 t_1}\\
\alpha &= \left(\mu_2 - \mu_1\right).
\end{aligned}
$$

3 Properties of Instrument

In a standard single barrier option, the barrier is observed for its entire life, from time $t = 0$ to time $t = T_{exp} = T_2$, with the value of the option determined accordingly. In a partial barrier option, the period of observation is a subset of the life of the option. Consider times t_1 such that $0 < t_1 < T_2$.

In a late start single partial barrier option, the barrier is active for the time frame $t_1 \leq t \leq T_2$, with two existing approaches to value the option:

1) the option is knocked in or out *only* when the barrier is touched or crossed, or

2) the barrier is regarded as a division of the outcome space (giving an area where the option is 'in' and one where it is 'out'), with evaluation depending on the position of the underlying in the outcome space during the life of the option.

Changes in the underlying during $0 < t < t_1$ do not affect the value of the option.

We consider only case 2).

Formulae for pricing this type of options were originally published by Heynen and Kat (1994).[1]

Bibliography

Espen Gaarder Haug. *The Complete Guide To Option Pricing Formulas.* McGraw Hill, New York, 2nd edition, 2007.

Ronald C. Heynen and Harry M. Kat. Partial barrier options. *The Journal of Financial Engineering*, 3(3/4):253–274, September/December 1994.

[1] Haug (2007) p.160 4.17.4 *Partial-Time Single-Asset Barrier Options*